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THE FUNCTIONAL EQUATIONS OF UNDISCOUNTED MARKOV RENEWAL PROGRAMMING

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The Functional Equations of Undiscounted Markov Renewal Programming *)

by

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ABSTRACT

This paper investigates the solutions to the functional equations that arise a.o. in the Undiscounted Markov Renewal Programming. We show that the solution set is a connected, though non-convex set whose members are unique up to n^* constants, characterize n^* and show that these n^* degrees of freedom are locally rather than globally independent.

Our results generalize those obtained in ROMANOVSKY [15] where another approach is followed for a special class of discrete time Markov Decision Processes.

Basically our methods involve the set of randomized policies. We first study the sets of pure and randomized maximal-gain policies, as well as the set of states that are recurrent under some maximal-gain policy.

KEY WORDS & PHRASES: *Markov Renewal Programs, average return optimality, functional equations, fixed points*

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*) This paper is not for review; it is meant for publication elsewhere.

I. INTRODUCTION

This paper investigates the solutions (g,v) to the 2N functional equations:

$$(1.1)\qquad g_i = \max_{k\in K(i)} \sum_{j=1}^{N} P_{ij}^k g_j, \qquad v = 1,\ldots,N$$

$$(1.2)\qquad v_i = \max_{k\in L(i)} \left[q_i^k - \sum_{j=1}^{N} H_{ij}^k g_j + \sum_{j=1}^{N} P_{ij}^k v_j\right], \qquad v = 1,\ldots,N,$$

where

$$L(i) = \left\{k \in K(i) \mid g_i = \sum_{j=1}^{N} P_{ij}^k g_j\right\}.$$

The K(i) are given finite sets and the $q_i^k, P_{ij}^k, H_{ij}^k$ are given arrays with $P_{ij}^k, H_{ij}^k \geq 0$ for all i,j,k; $\sum_{j=1}^N P_{ij}^k = 1$ and $\sum_{j=1}^N H_{ij}^k = T_i^k > 0$, for all i,k. Also we assume property P to be stated below.

For the special cases $H_{ij}^k = P_{ij}^k.\tau_{ij}^k$ with $\tau_{ij}^k \geq 0$ and $H_{ij}^k = \delta_{ij}$, the functional equations arise in Markov Decision Theory with $\Omega = \{1,\ldots,N\}$ as state space, q_i^k as the one-step expected reward, P_{ij}^k the transition probability to state j and T_i^k the expected holding time, when alternative k is chosen in state i (cf. BELLMAN [1,2], BLACKWELL [3], HOWARD [9,10], DE CANI [5], JEWELL [11], DENARDO & FOX [7], DENARDO [6], DERMAN [8], SCHWEITZER [16,17,18]).

The solution to (1.1) and (1.2) is not unique, although g is uniquely determined. The purpose of this paper is to characterize

$$V = \{v \in E^N \mid v \text{ satisfies } (1.2)\}.$$

We show that V is a connected, though non-convex set whose members are unique up to n^* constants, characterize n^*, and show that these n^* degrees of freedom are locally rather than globally independent.

Our results generalize those obtained in ROMANOVSKY [15] where another approach is followed for a special class of discrete time Markov Decision Processes (MDP's).

Basically our methods involve the set of randomized policies. We first study the sets S_{PMG} and S_{RMG} of pure and randomized maximal-gain policies, and characterize the set R^* of states that are recurrent under some maximal gain policy. In section 2 we give the notations and some preliminaries. In section 3 we characterize the sets S_{RMG} and R^*. The properties of V are studied in section 4, while in section 5 the n^* degrees of freedom are characterized. Finally, in section 6 some remarks are made with respect to a triangular decomposition of the set V.

II. NOTATIONS AND PRELIMINARIES

A *(stationary) randomized policy* f is a tableau $[f_{ik}]$ satisfying $f_{ik} \geq 0$ and $\sum_{k\in K(i)} f_{ik} = 1$ for all $i \in \Omega$. In the Markov decision model f_{ik} denotes the probability that the k^{th} alternative is chosen when entering state i.

We let S_R denote the set of all randomized policies and S_P the subset of all *pure* (non-randomized) policies, i.e. for $f \in S_P$ each $f_{ik} = 0$ or 1. For $f \in S_P$, we use the notation $f^{\#} = (\beta_1,\ldots,\beta_N)$ where $\beta_i \in K(i)$ denotes the single alternative used in state i.

Associated with each $f \in S_R$ are N-component "reward" vector q(f) and "holding time" vector T(f), and two matrices P(f) and H(f):

$$q(f)_i = \sum_{k\in K(i)} f_{ik}q_i^k; \qquad T(f)_i = \sum_{k\in K(i)} f_{ik}T_i^k$$

$$P(f)_{ij} = \sum_{k\in K(i)} f_{ik}P_{ij}^k; \qquad H(f)_{ij} = \sum_{k\in K(i)} f_{ik}.H_{ij}^k.$$

Note that P(f) is a stochastic matrix. For any $f \in S_R$, define the stochastic matrix $\Pi(f)$ as the Cesaro limit of the sequence $\{P^n(f)\}_{n=1}^{\infty}$ and define the *fundamental* matrix Z(f) as $[I - P(f) + \Pi(f)]^{-1}$. These matrices always exist and have the following properties (cf. [3],[12]):

$$\Pi(f) = P(f)\Pi(f) = \Pi(f)P(f) = \Pi(f)^2 = \Pi(f)Z(f) = Z(f)\Pi(f) \tag{2.1}$$

$$[I - P(f)]Z(f) = Z(f)[I - P(f)] = I - \Pi(f) \tag{2.2}$$

$$Z(f) = I + \lim_{a\uparrow 1} \sum_{n=0}^{\infty} a^n[P(f)^n - \Pi(f)]. \tag{2.3}$$

Denote by n(f) the number of subchains (closed, irreducible sets of states) for P(f). Then:

$$(2.4)\qquad \Pi(f)_{ij} = \sum_{m=1}^{n(f)} \phi_i^m(f)\pi_j^m(f), \qquad 1 \le ij \le N$$

where $\pi^m(f)$ is the unique equilibrium distribution of P(f) on the m^{th} subchain $C^m(f)$, and $\phi_i^m(f)$ is the probability of absorbtion in $C^m(f)$, starting from state i (cf. [6] and [18]). Observe $\sum_i \pi_i^m(f) = 1$ and $\pi^m(f)P(f) = \pi^m(f)$.

Let $R(f) = \{j \mid \Pi(f)_{jj} > 0\}$, i.e. R(f) is the set of recurrent states for P(f). Note that $\phi^m(f) = P(f)\phi^m(f)$ for all m and that the vecotrs $\phi^m(f)$ are linearly independent. Since any solution to $P(f)x = x$ satisfies $\Pi(f)x = x$ and the rank of $[I - \Pi(f)]$ is $N \div n(f)$, it easily follows that the solution set of $P(f)x = x$ is given by:

$$(2.5)\qquad x = \sum_{m=1}^{n(f)} a_m\phi^m(f)$$

with $a_1,\ldots,a_{n(f)}$ arbitrary scalars.

LEMMA 2.1. *Fix* $f \in S_R$, *and let the vector* b *satisfy* $\Pi(f)b = 0$. *Then* $[I - P(f)]x \ge b$, *implies* $x \ge Z(f)b + \Pi(f)x$, *where in both inequalities the equality sign holds for each component* $i \in R(f)$.

PROOF. Multiplying $[I - P(f)]x \ge b$ by $\Pi(f) \ge 0$, yields $0 = \Pi(f)[I - P(f)]x \ge$ $\ge \Pi(f)b = 0$, implying that the former inequality is a strict equality for components $i \in R(f)$. Using this and the fact that as a result of (2.3), for $j \notin R(f)$, $Z(f)_{ij} \ge 0$ for all i, with $Z(f)_{ij} = 0$ when $i \in R(f)$, we get the desired result by multiplying $[I - P(f)]x \ge b$ by Z(f) and invoking (2.2). □

LEMMA 2.2. *Let* $f \in S_R$, *and let* $C^m(f)$ *be any subchain of* P(f). *Take any* $i \in C^m(f)$ *and any* $k \in K(i)$ *with* $f_{ik} > 0$. *Then there exists a pure policy* h *such that* (a) $h_{ik} = 1$, (b) *for every* (j,r) $h_{jr} = 1$ *only if* $f_{jr} > 0$, (c) i *belongs to a subchain* C *of* P(h) *which is contained within* $C^m(f)$ *and* (d) $R(h) \subseteq R(f)$.

PROOF. Since $C^m(f)$ is closed for P(f), it is closed for any h meeting (b). Now, let $h_{ik} = 1$. If $C^m(f) = \{i\}$, condition (c) is satisfied. Otherwise, let Δ initially be equal to $\{i\}$. Define $\bar{\Delta} = C^m(f)\backslash\Delta$. Next the following step is performed:

Choose a state $j \in \bar{\Delta}$ and an alternative r such that $f_{jr} > 0$ and $P^r_{jt} > 0$ for some $t \in \Delta$, transfer j from $\bar{\Delta}$ to Δ, and define $h_{jr} = 1$. Clearly, such a j and r can be found, since all states in $C^m(f)$ communicate under $P(f)$. Repeat this step for the new Δ and $\bar{\Delta}$, until $\bar{\Delta}$ is empty. This construction shows that under policy h, state i can be reached from any state in $C^m(f)\backslash\{i\}$. Together this and the fact that $C^m(f)$ is closed under $P(h)$, imply *condition* (c). *Condition* (d) trivially holds if $\Omega = R(f)$. Otherwise, let Γ initially be equal to $R(f)$ and define $\bar{\Gamma} = \Omega - \Gamma$. Choose a state $t_0 \in \bar{\Gamma}$ and a path $\{t_0,t_1,\ldots,t_n\}$ such that $P(f)_{t_\ell t_{\ell+1}} > 0$ for $\ell = 0,\ldots$ $\ldots,n-1$ and $t_n \in \Gamma$. Such a path clearly exists, since t_0 is transient under $P(f)$ and $\Gamma \supseteq R(f)$. Transfer $\{t_0,\ldots,t_{n-1}\}$ from $\bar{\Gamma}$ to Γ and define for $\ell = 0,\ldots,n-1$ $h_{t_\ell r} = 1$ for some r with $f_{t_\ell r} > 0$ and $P^r_{t_\ell t_{\ell+1}} > 0$. Repeat this step until $\bar{\Gamma}$ is empty. Finally, for $j \in R(f) - C^m(f)$, define $h_{jr} = 1$ for some r, with $f_{jr} > 0$ and observe that *condition* (b) holds for all $j \in \Omega$. This completes the proof. □

In the remainder of the paper, we assume that property P holds.

P: If f is any pure policy and $C^m(f)$ is any subchain of $P(f)$, then $i \in C^m(f)$ implies $H(f)_{ij} = 0$ for $j \notin C^m(f)$.

This property is satisfied for both the Markov Renewal Programs (MRP's) with $H^k_{ij} = P^k_{ij}\tau^k_{ij}$ and the discrete time model with $H^k_{ij} = \delta_{ij}$. Using the previous lemma, one easily verifies that if property P holds for all pure policies, it holds for all randomized policies.

<u>LEMMA 2.3</u>. (*Gain and Relative Value Vectors*).
Fix $f \in S_R$. *The general solution to the equations*

$$(2.6) \qquad (a) \quad g = P(f)g, \qquad (b) \quad v = q(f) - H(f)g + P(f)v$$

is given by

$$(2.7) \qquad g_i = g(f)_i = \sum_{m=1}^{n(f)} \phi^m_i(f) g^m(f),$$

with

$$g^m(f) = \langle\pi^m(f),q(f)\rangle/\langle\pi^m(f),T(f)\rangle$$

and

$$(2.8) \qquad v_i = Z(f)[q(f) - H(f)g]_i + \sum_{m=1}^{n(f)} a_m\phi_i^m(f),$$

with $a_1,\ldots,a_{n(f)}$ *arbitrary scalars.*

PROOF. Note that multiplication of (2.6)(b) by $\Pi(f)$ leads to :

$$(2.9) \qquad \Pi(f)[q(f) - H(f)g] = 0.$$

Using property P, it follows from the proof of lemma 1 of [6] that $g(f)$ is the unique solution to (2.6)(a) and (2.9). Hence, any solution (g,v) to (2.6) has $g = g(f)$. Using (2.2) one next verifies by mere insertion that $(g=g(f),v=Z(f)[q(f)-H(f)g(f)])$ satisfy (2.6). Finally (2.8) follows from (2.5), since (2.6)(b) is a linear system of equations with $Z(f)[q(f) - H(f)g(f)]$ as a particular solution. □

The unique solution $g(f)$ to (2.6) will be called the *gain rate vector*, and $g^m(f)$ the gain rate of the subchain $C^m(f)$. A solution v to (2.6) will be called a *relative-value vector* and denoted by $v(f)$.

In the remainder, we will refer to the following example:

EXAMPLE 1. $N = 4$, $K(1) = K(2) = \{1\}$; $K(3) = \{1,2\}$; $H_{ij}^k = \delta_{ij}$ for all i,j,k.

i	k	p_{i1}^k	p_{i2}^k	p_{i3}^k	p_{i4}^k	q_i^k
1	1	0	1	0	0	0
2	1	1	0	0	0	0
3	1	1	0	0	0	$q_3^1 \le 0$
3	2	0	0	1	0	0
4	1	.4	.4	.2	0	0
4	2	.8	.2	0	0	0

Using (3.1) and theorem 3.1, part (c) one verifies that

$$V = \{v^* \in E^4 \mid v_1^* = v_2^*;\ v_3^* \geq q_3^1 + v_1^*;\ v_4^* = \max[.8v_1^* + .2v_3^*;\ v_1^*]\}$$

Observe that V is non-convex. Note furthermore, that for $f \in S_{RMG}$, if f makes unwise decisions in states in $\Omega - R(f)$, then there do not necessarily exist additive constants such that $v(f) \in V$ (cf. theorem 3 of [17] and our theorem 4.1 part (b)). Take the above example with pure policy $f^{\#} = (1,1,1,1)$ with P(f) unichained, and $v(f) = (0\ 0\ q_3^1\ .2q_3^1) + a(1\ 1\ 1\ 1) \notin V$ for any choice of the additive contant a.

In addition, we observe that the Policy Interation Algorithm (PIA) (cf. [5], [7], [11]) is not guaranteed to converge, if unwise choices for the additive constants in (2.8) are made. Consider the above example with $q_3^1 < 0$, $f^{1\#} = (1,1,2,1)$ and $f^{2\#} = (1,1,2,2)$. Then $v(f^1) = \lambda[1\ 1\ 0\ .8] + \mu[0\ 0\ 1\ .2]$ and $v(f^2) = \nu[1\ 1\ 0\ 1] + \rho[0\ 0\ 1\ 0]$, for arbitrary λ,μ,ν,ρ. Choosing $q_3^1 + \lambda \leq \mu < \lambda$ and $\rho > \nu$, f^1 and f^2 follow each other in the PIA. Fortunately, PIA cycling can be prevented by preserving the old additive constant in a subchain, whenever the subchain is preserved (see also [20]).

III. PROPERTIES OF MAXIMAL GAIN POLICIES

We first introduce some notations. Define the *maximal gain rate*

$$(3.1)\qquad g_i^* = \sup_{f \in S_R} g(f)_i, \qquad i = 1,\ldots,N.$$

For any $v \in V$, define

$$b(v)_i^k = q_i^k - \sum_j H_{ij}^k g_j^* + \sum_j P_{ij}^k v_j - v_i,$$

and

$$b(v,f)_i = \sum_{k \in K(i)} b(v)_i^k = [q(f) - H(f)\overset{*}{g} + P(f)v - v]_i$$

Since g(f) can be interpreted as the average reward of f for a MRP with transition probabilities P_{ij}^k, one-step expected rewards q_i^k, and holding times T_i^k, we know from DERMAN [8] that there exists a pure policy that attains the N suprema in (3.2) simultaneously. Hence $g_i^* = \max_{f \in S_P} g(f)_i$.

Accordingly define:

$$S_{PMG} = \{f \in S_P \mid g(f) = g^*\}$$

and

$$S_{RMG} = \{f \in S_R \mid g(f) = g^*\}.$$

Finally, let:

$$w_i^* = \max_{f \in S_{PMG}} Z(f)[q(f) - H(f)g^*]_i.$$

THEOREM 3.1. *(Properties of Maximal-Gain Policies).*

(a) $f \in S_{RMG}$ *if and only if* $g^* = P(f)g^*$ *and* $\Pi(f)[q(f) - H(f)g^*] = 0$.

(b) *The functional equations* (1.1) *and* (1.2) *always have the solution* $g = g^*$, $v = w^*$. *Hence* V *is non-empty. Also, there exists a policy* $f \in S_{PMG}$ *such that* $w^* = Z(f)[q(f) - H(f)g^*]$.

(c) *In any solution* (g,v) *of the functional equations* (1.1) *and* (1.2) $g = g^*$, *hence* g *and each* L(i) *is unique.*

(d) *If* f *is any policy, and if* C *is any subchain of* P(f) *then* $g_i^* =$ *constant,* $i \in C$.

(e) *If* $v \in V$, *then* $\max_{k \in L(i)} b(v)_i^k = 0$, *for every* i. *Let* $f \in S_R$.

  (1) *Suppose that* $k \in L(i)$ *for each* (i,k) *with* $f_{ik} > 0$ *and that for some* $v \in V$, $b(v)_i^k = 0$ *for each* (i,k) *with* $i \in R(f)$ *and* $f_{ik} > 0$. *Then* $f \in S_{RMG}$.

  (2) *Conversely, if* $f \in S_{RMG}$, *then for each* $i = 1,\dots,N$ $f_{ik} > 0$ *implies* $k \in L(i)$, *and for* $i \in R(f)$, $f_{ik} > 0$ *implies* $b(v)_i^k = 0$ *for all* $v \in V$.

PROOF.

(a) From the proof of lemma 2.3 we know that g(f) is the unique solution to the equations $g = P(f)g$ and (2.9).

(b) Invoking the above mentioned interpretation of g^*, we know from theorem 1 in DENARDO & FOX [7] that $g_i^* = \max_k \sum_j p_{ij}^k g_j^*$. Consider the discrete time decision model with $\bar{K}(i) = L(i) = \{k \mid g_i^* = \sum_j P_{ij}^k g_j^*\}$, $\bar{P}_{ij}^k = P_{ij}^k$ and $\bar{q}_i^k = q_i^k - \sum_j H_{ij}^k g_j^*$.

Note that in this model, each policy has $\bar{g}(f) \le 0$. Moreover, it follows from part (a) that $\bar{g}(f) = 0$ if and only if $f \in S_{RMG}$. Hence the discrete time model has $\bar{g}^* = 0$ and, with $\bar{S}_{PMG} = \{f \in X_{i=1}^N \bar{K}(i) \mid \bar{g}(f) = \bar{g}^* = 0\}$, we have:

$$\max_{f \in S_{PMG}} Z(f)[q(f) - H(f)g^*]_i = \max_{f \in \bar{S}_{PMG}} Z(f)\{\bar{q}(f) - \bar{g}^*\}_i, \quad \text{for } i = 1,\ldots,N.$$

Use theorem 4 of [3] in order to prove the existence of a policy $f \in S_{PMG}$ for which $w^* = Z(f)[q(f) - H(f)g^*]$ as well as the fact that w^* satisfies (1.2).

(c) Fix a solution (g,v) to (1.1) and (1.2). Using property P, a minor modification of the proof of lemma 4 of [7], shows that $g \geq g(f)$ for all $f \in S_P$ with equality for any f^o, such that $f^o_{ik} = 1$ for some k maximizing (1.1) and (1.2). Hence $g = g^*$.

(d) Since g^* satisfies (1.1), we have $P(f)g^* \leq g^*$ for all $f \in S_R$. The assertion then follows from lemma 2-a in [7].

(e) The first result follows from the very definition of $b(v)_i^k$

(1) From the definition of $b(v)_i^k$, we have $v_i - \sum_j P(f)_{ij}v_j = q(f)_i - \sum_j H(f)_{ij}g^*_j$ for $i \in R(f)$. Multiplying this equation with $\Pi(f)_{ki}$ and summing over i, we obtain $\Pi(f)[q(f) - H(f)g^*] = 0$. Use this, and $g^* = P(f)g^*$ in order to apply part (d).

(2) If $f \in S_{RMG}$, $g^* = P(f)g^*$ follows from part (d). Hence $f_{ik} > 0$ implies $k \in L(i)$ and $b(v)_i^k \leq 0$. So $b(v,f) \leq 0$, for any $v \in V$. Since we know from part (d) that $\Pi(f)b(v,f) = 0$ for $f \in S_{RMG}$, it follows that for $j \in R(f)$, $b(v,f)_j = 0$, i.e. $f_{ik} > 0$ implies $b(v)_j^k = 0$. □

Define next

(3.2) $\qquad R^* = \{i \mid i \in R(f) \text{ for some policy } f \in S_{RMG}\}.$

The following theorem gives a characterization of this set, which plays a basic part in the remainder of this paper.

<u>THEOREM 3.2</u>. (*Characterization of* R^*).

(a) $R^* = \{i \mid i \in R(f)$ *for some* $f \in S_{PMG}\}$.

(b) *The set* $\{f \in S_{RMG} \mid R(f) = R^*\}$ *is not empty.*

(c) *Define* $n^* = \min\{n(f) \mid f \in S_{RMG}$ *with* $R(f) = R^*\}$ *and* $S^*_{RMG} =$
$= \{f \in S_{RMG} \mid R(f) = R^*$ *and* $n(f) = n^*\}$. *Fix* $f^* \in S^*_{RMG}$. *Any subchain of any* $f \in S_{RMG}$ *is contained within a subchain of* $P(f^*)$.

(d) *All* $f^* \in S^*_{RMG}$ *have the same collection of subchains* $\{R^{*\alpha}, \alpha = 1,\dots,n^*\}$.

(e) *For any* $1 \le \alpha \le n^*$, $g^*_i = g^{*\alpha}$ *(say) for all* $i \in R^{*\alpha}$.

(f) *Let* $R^{(1)},\dots,R^{(m)}$ *be disjoint sets of states such that*
   (1) *if* C *is a subchain of some* $f \in S_{RMG}$, *then* $C \subseteq R^{(k)}$ *for some* k, $1 \le k \le m$;
   (2) *there exists a* $f^* \in S_{RMG}$ *with* m *subchains* $\{R^{(k)}\}_{k=1}^{m}$.
   Then $m = n^*$ *and after renumbering* $R^{(\alpha)} = R^{*\alpha}$ *for* $\alpha = 1,\dots,n^*$.

PROOF.

(a) Fix a state i, and a $f \in S_{RMG}$ such that $i \in R(f)$. Consider a policy h satisfying the conditions (a), (b), (c) and (d) of lemma 2.2. Using theorem 3.1. part (e), one verifies that $h \in S_{PMG}$, and $i \in R(h)$. Thus the right-hand side of (a) is included in R^* and the reversed inclusion is immediate.

(b) Fix an enumeration $f^1,\dots,F^M$ of S_{PMG}. For any $i \in R^*$, let $A_i = \{r \mid i \in R(f^r)\}$. Consider the following equivalence relation on $\mathcal{C} = \{C^m(f^r) \mid 1 \le r \le M;\ 1 \le m \le n(f^r)\}$:

Let $C \sim C'$, if there exists $\{C^{(1)}=C, C^{(2)},\dots,C^{(n)}=C'\}$ with $C^{(i)} \in \mathcal{C}$ and $C^{(i)} \cap C^{(i+1)} \neq \emptyset$ for $i = 1,\dots,n-1$.

Let f^* satisfy: (1) $\{k \mid f^*_{ik} > 0\} = \cup_{r \in A_i}\{k \mid f^r_{ik} > 0\}$ for $i \in R^*$; (2) $\{k \mid f^*_{ik} > 0\} = L(i)$ for $i \in \Omega - R^*$. Using theorem 3.1 part (e) one verifies that $f^* \in S_{RMG}$.

Clearly, the equivalence classes are the subchains of $P(f^*)$ since they are closed under $P(f^*)$ and since the states belonging to a same equivalence class communicate with each other. Hence, $R^* = R(f^*)$.

(c) Assume $P(f)$ has a subchain $C^m(f)$ that intersects say R^{*1} and R^{*2}. Then a policy f^{**} with $\{k \mid f^{**}_{ik} > 0\} = \{k \mid f^*_{ik} > 0\}$ and $\{k \mid f^{**}_{ik} > 0\} = \{k \mid f_{ik} > 0\} \cup \{k \mid f^*_{ik} > 0\}$ otherwise, is maximal gain, has $R(f^{**}) = R^*$, and its number of subchains is at most $n^* - 1$, since the states of R^{*1} and R^{*2} communicate with each other under $P(f^{**})$. This contradicts the minimality of n^*.

(d) For all $f^*, f^{**} \in S^*_{RMG}$, part (c) implies each $C^\alpha(f^*) \subseteq$ some $C^\beta(f^{**})$, and each $C^\beta(f^{**}) \subseteq C^\alpha(f^*)$.

(e) Combine part (d) with part (c) of theorem 3.1.

(f) Apply property (1) to conclude $R^{*\alpha} \subseteq R^{(k(\alpha))}$. Apply part (c) and property (2) to conclude $R^{(k(\alpha))} \subseteq R^{*\alpha}$. □

REMARK 1. Note that as a result of part (f) of the above theorem, the policy f^* that was constructed in the proof of part (b), belongs to S^*_{RMG}. Verify that the definition of f^* implies any subchain of a maximal gain policy to be contained in a subchain of $P(f^*)$.

A finite procedure for calculating R^*, n^*, the $R^{*\alpha}$ and a $f^* \in S^*_{RMG}$ is therefore as follows: use the PIA to find g^* and a $v \in V$. Compute $S_P(v) = X_{i=1}^N \{k \in L(i) \mid b(v)_i^k = 0\} = \{f \in S_P \mid f$ achieves the 2N maxima in (1.1) and (1.2)$\} \subseteq S_{PMG}$. Part (a) of theorem 3.2 in combination with part (a) of theorem 3.1 establish $R^* = \{i \mid i \in R(f),\ f \in S_P(v)\}$. Determine $R^{*\alpha}$ as the equivalence classes of the set of subchains of policies belonging to $S_P(v)$ (cf. proof of theorem 3.1 part (b) and remark 1). Finally, define f^* by $\{k \mid f^*_{ik} > 0\} = L(i)$ for $i \in \Omega - R^*$, and $\{k \mid f^*_{ik} > 0\} = \{k \in L(i) \mid b(v)_i^k = 0,\ \sum_{j \in R^{*\alpha}} P^k_{ij} = 1\}$ for $i \in R^{*\alpha}$ $(\alpha = 1,\ldots,n^*)$.

VI. PROPERTIES OF V

Some basic properties of V are given by:

THEOREM 4.1. (*Basic Properties of* V).

(a) V *is closed an unbounded, as* $v \in V$ *implies* $v + a_1\underline{1} + a_2 g^* \in V$, *for any scalars* a_1, a_2 *(where* $\underline{1}$ *is the* N*-vector with all coordinates unitary).*

(b) *(Maximality of relative values.) For any* $v^* \in V$ *and* $f \in S_{RMG}$, *it is possible to choose the* $n(f)$ *additive constants in* $v(f)$ *such that* $v^* \geq v(f)$ *with equality for components in* $R(f)$.

(c) *(Cf.* [2],[16].*)* $v \in V$, *if and only if*

$$(4.1) \qquad v_i = \max_{f \in S_{PMG}} \{Z(f)[q(f) - H(f)g^*]_i + \Pi(f)v_i\} \qquad i = 1,\ldots,N.$$

In addition, if $v \in V$*, then a policy* $f \in S_{PMG}$ *achieves all* N *maxima in* (4.1) *if and only if it achieves the* 2N *maxima in* (1.1) *and* (1.2).

PROOF.

(a) Immediate to verify.

(b) Choose in (2.8) $a_m = \langle \pi^m(f), v^* \rangle$. From part (e) of theorem 3.1, it follows that $\{k \mid f_{ik} > 0\} \subseteq L(i)$ for each i, hence $v^* \geq q(f) - H(f)g^* + P(f)v^*$, which implies, using (2.9), lemma 2.1, (2.4) and (2.8):

$$v^* \geq Z(f)[q(f) - H(f)g^*] + \Pi(f)v^* =$$

$$= Z(f)[q(f) - H(f)g^*] + \sum_{m=1}^{n(f)} a_m \phi^m(f) = v(f)$$

with equality for components in R(f).

(c) First assume $v \in V$. In part (b) we proved that for any $f \in S_{PMG}$, $v \geq Z(f)[q(f) - H(f)g^*] + \Pi(f)v$, with strict equality for $f \in S_P(v)$. Hence, $v \in V$ implies (4.1) and any policy achieving the 2N maxima in (1.1) and (1.2) acheives all N maxima in (4.1).

Conversely, if v satisfies (4.1), we define:

$$(4.2) \qquad \tilde{v}_i = \max_{k \in L(i)} [q_i^k - \sum_j H_{ij}^k g_j^* + \sum_j P_{ij}^k v_j],$$

and show both $\tilde{v} \geq v$ and $\tilde{v} \leq v$, hence $\tilde{v} = v \in V$.

For any $f \in S_{PMG}$, $f_{ik} = 1$ implies $k \in L(i)$ by theorem 3.1 part (e); hence, using (4.1), (2.2) and (2.9):

$$\tilde{v} \geq q(f) - H(f)g^* + P(f)v \geq [I + P(f)Z(f)][q(f) - H(f)g^*] + \Pi(f)v =$$

$$= Z(f)[q(F) - H(f)g^*] + \Pi(f)v, \qquad f \in S_{PMG}.$$

This implies $\tilde{v} \geq v$. Let h denote a pure policy in $X_{i=1}^N L(i)$, achieving all maxima in (4.2). Then:

$$(4.3) \qquad v_i \leq \tilde{v}_i = [q(h) - H(h)g^* + P(h)v]_i.$$

Multiply (4.3) with $\Pi(h) \geq 0$ in order to get $0 \leq \Pi(h)[q(h) - H(h)g^*] \leq 0$, the latter inequality following from (2.9) and $g(h) \leq g^*$. Hence (4.3) is an equality for $i \in R(h)$, and so $h \in S_{PMG}$, by part (e) of theorem 3.1.

Using lemma 2.1, (4.3) implies $v \le Z(h)[q(h) - H(h)g^*] + \Pi(h)v$. Insert on the right-hand side of (4.2) and use $\Pi(h)[q(h) - H(h)g^*] = 0$, to obtain:

$$\tilde{v} \le [I + P(h)Z(h)][q(h) - H(h)g^*] + \Pi(h)v =$$

$$= Z(h)[q(h) - H(h)g^*] + \Pi(h)v \le$$

$$\le \max_{f \in S_{PMG}} \{Z(f)[q(f) - H(f)g^*] + \Pi(f)v\} = v.$$

Finally, if $h \in S_{PMG}$ achieves the N maxima in (4.1), multiply the equality portion of this inequality with $Z(h)^{-1}$ to show that it achieves the N maxima in (1.2), as well as the N maxima in (1.1), since $h_{ik} = 1$ implies $k \in L(i)$. This completes the proof. □

Since for $f \in S_{RMG}$, $\Pi(f)_{ij} = 0$ if $j \notin R^*$, we have by part (c) of theorem 4.1 that $v \in V$ if and only if

$$v_i = \max_{f \in S_{PMG}} \{Z(f)[q(f) - H(f)g^*]_i + \sum_{j \in R^*} \Pi(f)_{ij} v_j\}, \qquad i \in R^* \tag{4.4}$$

$$v_i = \max_{f \in S_{PMG}} \{Z(f)[q(f) - H(f)g^*]_i + \sum_{j \in R^*} \Pi(f)_{ij} v_j\}. \qquad i \in \Omega \backslash R^*. \tag{4.5}$$

Observe that (4.4) involves only $(v_i | i \in R^*)$ and can be studied in isolation. The $(v_i | i \in \Omega\backslash R^*)$ are uniquely determined via (4.5), for any $(v_i | i \in R^*)$. Define now

$$V^R = \{(v_i | i \in R^*);\ v_i \text{ satisfy (4.4) for all } i \in R^*\}. \tag{4.6}$$

<u>THEOREM 4.2.</u>

(a)

$$V^R = \{(v_i | i \in R^*);\ v_i \ge Z(f)[q(f) - H(f)g^*]_i + \sum_{j \in R^*} \Pi(f)_{ij} v_j, \textit{ for all } i \in R^*,\ f \in S_{PMG}\}. \tag{4.7}$$

Hence, V^R *is a closed, convex polyhedral set.*

(b) V *is connected.*

PROOF.

(a) Clearly, V^R is contained within the polyhedron, that is defined in the right side of (4.7). Conversely fix $i \in R^*$ and $h \in S_{PMG}$ with $i \in R(h)$. Then, by multiplying the inequalities in (4.7) with $\Pi(h) \geq 0$, we obtain $v_i = Z(h)[q(h) - H(h)g^*]_i + \sum_{j \in R^*} \Pi(h)_{ij} v_j$; hence (4.4) holds.

(b) The assertion follows by showing that for any $v, \tilde{v} \in V$, the curve $\{\underline{v}(\lambda) \mid \lambda \in [0,1]\}$ with parameter representation: $v(\lambda)_i = \lambda v_i + (1-\lambda)\bar{v}_i$, $i \in R^*$ and $v(\lambda)_i = \max_{f \in S_{PMG}} \{Z(f)[q(f) - H(f)g^*]_i + \sum_{j \in R^*} \Pi(f)_{ij} v(\lambda)_j\}$ connects v with $\tilde{v}$, lies within V as a consequence of (4.5) and part (a), and is continuous, since all its components are continuous functions of λ. □

We already saw that V may not be convex. The following theorem gives a necessary and sufficient condition for the convexity of V.

THEOREM 4.3. V *is convex if and only if for each* $i \in \Omega - R^*$ *there exists an alternative* $k(i) \in L(i)$, *such that for all* $v \in V$:

$$(4.8) \qquad v_i = q_i^{k(i)} - \sum_j H_{ij}^{k(i)} g_j^* + \sum_j P_{ij}^{k(i)} v_j .$$

Moreover, V *is convex if and only if it is a polyhedron.*

PROOF. We first observe that for any $i \in R^*$, there is a $h \in S_{PMG}$, with $i \in R(h)$, hence by part (e) of theorem 3.1 there exists an alternative $k(i) \in L(i)$ with $b(v)_i^{k(i)} = 0$, for any $v \in V$. Thus (4.8) always holds for $i \in R^*$. Suppose it holds for $i \in \Omega - R^*$ as well. Then the functional equations are equivalent to the linear (in)equalities $b(v)_i^{k(i)} = 0$ for $i = 1,\ldots$ $\ldots,N$ and $b(v)_i^k \leq 0$ for $k \in L(i)\backslash\{k(i)\}$ and $i = 1,\ldots,N$. Hence V is a convex polyhedron.

Conversely, suppose V is convex. Assume to the contrary that there exists a state $i \in \Omega - R^*$ and a finite set of $v^{(m)}$'s in V, such that no $k \in L(i)$ achieves the maximum in (1.2) for all $v^{(m)}$. However, since V is convex, it is immediate to verify that a $k \in L(i)$ achieving the maximum in (1.2) for a positive convex combination $\bar{v}$ of the $v^{(m)}$'s, achieves the maximum in (1.2) for each $v^{(m)}$. □

REMARK 2. (4.8), hence convexity of V is trivially met if either (1) $R^* = \Omega$, (2) $L(i)$ is a singleton for each $i \in \Omega - R^*$, or (3) there is only one maximal gain policy.
In addition $n^* = 1$ is sufficient for the convexity of V as well. This follows by considering a $f^* \in S^*_{RMG}$. By theorem 4.2 part (b), we obtain that for each $v \in V$, there exists a relative value vector $v(f^*)$ such that $v_i = v(f^*)_i$, $i \in R^*$. $P(f^*)$ being unichained, it follows that $v(f^*)$ is unique up to a multiple of $\underline{1}$, hence $(v_i | i \in R^*)$ is unique up to an additive constant. Using (4.5), we conclude that $v \in V$ is unique up to a multiple of $\underline{1}$.

For discrete time Markovian decision processes, where $H^k_{ij} = \delta_{ij}$, the value-iteration equations take the form:

$$(4.9) \qquad v(n+1)_i = \max_{k \in K(i)} \{q^k_i + \sum_j P^k_{ij} v(n)j\},$$

with $v(0)$ a given vector.

It is well known that $\{v(n) - ng^*\}_{n=1}^{\infty}$ may fail to converge. In a forthcoming paper [19] it will be shown that there exists an integer J such that

$$u_i^{(r)} = \lim_{n \to \infty} \{v(nJ+r) - (nJ+r)g^*_i\}$$

exists for all i, with $u_i^{(r+J)} = u_i^{(r)}$ (previous proofs in [4] and [13] are both incorrect).

Accordingly, define $\bar{v}$ as the Cesaro-limit of the sequence $\{v(n) - ng^*\}_{n=1}^{\infty}$. Example 1 with $q_3^1 = 0$ and $v(0) = [1\ 0\ 1\ .6]$ shows that in general $\bar{v} \notin V$ ($v(2n)_1=1$; $v(2n+1)_1=0$; $v(2n)_2=0$; $v(2n+1)_2=1$; $v(n)_3=1$; $v(2n)_4=.8$; $\bar{v}=[.5\ .5\ 1\ .7] \notin V$).

The relation between v and V is as follows:

THEOREM 4.4.
(a) $\{\bar{v}_i \mid i \in R^*\} \in V^R$.
(b) *There exists a vector* $v \in V$, *such that* $v \le \bar{v}$ *with equality for components in* R^*.

PROOF. Note that for all $i \in \Omega$: $u_i^{(r+1)} = \max_{k \in K(i)} \{q^k_i - g^*_i + \sum_j P^k_{ij} u_j^{(r)}\}$, since for all n sufficiently large the maximizing alternatives in (4.9) belong to L(i) as observed in [4] and [13].

Since $v = \frac{1}{J}\sum_{r=0}^{J-1} u^{(r)}$, we obtain by averaging over $r = 0,\dots,J-1$:

$$\bar{v}_i \ge q_i^k - g_i^* + \sum_j p_{ij}^k \bar{v}_j, \qquad i = 1,\dots,N \text{ and } k \in K(i).$$

Take any $f \in S_{PMG}$ to obtain: $\bar{v} \ge q(f) - g^* + P(f)\bar{v}$, and hence, using lemma 2.1: $\bar{v} \ge Z(f)[q(f) - g^*] + \Pi(f)\bar{v}$, with equality for $i \in R(f)$. This implies: $\bar{v} \ge \max_{f\in S_{PMG}} \{Z(f)[q(f) - g^*] + \Pi(f)\bar{v}\}$ with equality for components in R^*. Using (4.4) and (4.5) we obtain that the vector v defined by (1) $v_i = \bar{v}_i$, $i \in R^*$ and (2) $v_i = \max_{f\in S_{PMG}} \{Z(f)[q(f) - g^*]_i + \sum_{j\in R^*} \Pi(f)_{ij} v_j\}$ for $i \in \Omega - R^*$, belongs to V with $v \le \bar{v}$ and equality for components in R^*. □

V. THE n^* DEGREES OF FREEDOM IN V

In this section we show that the convex polyhedral set V^R has dimension n^* and that its elements, and hence V, are fully determined by n^* parameters $(y_1,\dots,y_{n^*})$.

ROMANOVSKY [15] obtained the same result for the functional equations that arise in discrete time Markov models with $\underline{g}^* = \langle g^*\rangle \underline{1}$. In addition, as our methods involve the chain structure, a fuller characterization of the parameter space is possible.

The key observation is that any two vectors $v, \tilde{v} \in V$ have the property: $\tilde{v}_i - v_i = \text{constant} = y_\alpha$ for $i \in R^{*\alpha}$, $\alpha = 1,\dots,n^*$.

By fixing $v^o \in V$ and picking these n^* constants, one thus determines $(\tilde{v}_i \mid i\in R^*)$ and hence $\tilde{v}$ by (4.5) in terms of v^o. Hence, by fixing v^o, and sweeping out all permitted values of y, we sweep out all vectors $\tilde{v}$ in V. In particular (5.1) below shows that $\tilde{v}$ is a convex piecewise linear function in v.

<u>THEOREM 5.1</u>. *Let* $v \in V$. *The following are equivalent:*

(a) $v + x \in V$

(b) $x_i = \max_{k\in L(i)} [b(v)_i^k + \sum_j p_{ij}^k x_j]$, $\qquad i = 1,\dots,N$

(c) $x_i = \max_{f\in S_{PMG}} [Z(f)b(v,f) + \Pi(f)x]_i$, $\qquad i = 1,\dots,N$

(d) *there are* n* *constants* $y = (y_1,\dots,y_{n^*})$ *satisfying*

$$(5.1)\qquad x_i = \begin{cases} y_\alpha & i \in R^{*\alpha},\ \alpha = 1,\dots,n^* \\ \max\limits_{f\in S_{PMG}} \left[Z(f)b(v,f)_i + \sum\limits_{\beta=1}^{n^*}\left(\sum\limits_{j\in R^{*\beta}} \Pi(f)_{ij}\right)y_\beta\right], & i \in \Omega \setminus R^* \end{cases}$$

$$(5.2)\qquad y_\alpha \ge Z(f)b(v,f)_i + \sum_{\beta=1}^{n^*}\left(\sum_{j\in R^{*\beta}} \Pi(f)_{ij}\right)y_\beta,$$

$$\alpha = 1,\dots,n^*;\ i \in R^{*\alpha},\ f \in S_{PMG}.$$

PROOF.

(a) $\Leftrightarrow$ (b): b is the requirement that $v + x \in V$.

(a) $\Leftrightarrow$ (c): Cf. (4.1) and the definition of $b(v,f)$.

(a) $\Leftrightarrow$ (d): Take $f^* \in S^*_{RMG}$. As $v, v + x \in V$, we have from part (e) of theorem 3.1: $v_i = [q(f^*) - H(f^*)g^* + P(f^*)v]_i$ and $(v+x)_i = [q(f^*) - H(f^*)g^* + P(f^*)(v+x)]_i$ for all $i \in R^* = R(f^*)$. Subtraction yields: $x_i = [P(f^*)x]_i = [\Pi(f^*)x]_i = \langle\pi^\alpha(f^*),x\rangle$ for $i \in R^{*\alpha}$, which proves the first part of (5.1). Moreover, this implies the remainder of (d), using (4.4) and (4.5) and the definition of $b(v,f)$.

(d) $\Leftrightarrow$ (a): Use (4.4), (4.5) and the definition of $b(v,f)$. □

Fix $v \in V$. Define the set of allowed constants

$$Y(v) = \{y \in E^{n^*} \mid y \text{ satisfies } (5.2)\}.$$

The following theorem is obvious from the definition of $Y(v)$, theorem 4.1 part (a) and the fact that:

$$(5.3)\qquad Z(f)b(v,f) \le 0 \qquad\qquad \text{for all } f \in S_{PMG}.$$

(5.3) follows from lemma 2.1, with $x = 0$, using $b(v,f) \le 0$ and $\Pi(f)b(v,f) = 0$ (cf. theorem 3.1 part (d) and (e)).

THEOREM 5.2. *For any* $v \in V$, $Y(v)$ *is a closed, convex polyhedral set containing* $y = 0$, *(i.e.* $\lambda y \in V$, *for* $\lambda \in [0,1]$ *if* $y \in Y(v)$*).*

Furthermore, $Y(v)$ *is unbounded as* $[y_\alpha] \in Y(v)$, *implies* $[y_\alpha + c_1 + c_2 g^{*\alpha}] \in Y(v)$, *for any scalars* c_1, c_2.

Clearly, by (5.3), (5.2) is automatically satisfied for (α,i,f) with $\sum_{j\in R^{*\alpha}} \Pi(f)_{ij} = 1$. We accordingly define:

$$\tilde{K}(\alpha) = \{(i,f) \mid i \in R^{*\alpha},\ f \in S_{PMG},\ \sum_{j\in R^{*\alpha}} \Pi(f)_{ij} < 1\},\ \alpha = 1,\dots,n^*,$$

and make the partition $\{1,2,\dots,n^*\} = E \cup F$, where $E = \{\alpha \mid \tilde{K}(\alpha) = \emptyset\}$, $F = \{\alpha \mid \tilde{K}(\alpha) \neq \emptyset\}$.

For $\xi = (i,f) \in \tilde{K}(\alpha)$, define

$$\tilde{q}^{\xi}_{\alpha} = [Z(f)b(v,f)]_i, \qquad \text{and} \qquad \tilde{P}^{\xi}_{\alpha\beta} = \sum_{j\in R^{*\beta}} \Pi(f)_{ij}.$$

Note that $\tilde{q}^{\xi}_{\alpha} \le 0$, $\tilde{P}^{\xi}_{\alpha\beta} \ge 0$, $\sum_{\beta=1}^{n^*} \tilde{P}^{\xi}_{\alpha\beta} = 1$, $\tilde{P}^{\xi}_{\alpha\alpha} < 1$ for all $\alpha \in F$, and $\xi \in \tilde{K}(\alpha)$. Then $Y(v)$ consists of all $y \in E^{n^*}$ satisfying

$$(5.4) \qquad y_{\alpha} \ge \tilde{q}^{\xi}_{\alpha} + \sum_{\beta=1}^{n^*} \tilde{P}^{\xi}_{\alpha\beta} y_{\beta}, \qquad \alpha \in F,\ \xi \in K(\alpha).$$

The following theorem expresses that $(y_{\alpha} | \alpha\in E)$ are fully independent degrees of freedom:

THEOREM 5.3.
(a) *Let* $(y_{\alpha}|\alpha\in E)$ *be arbitrary. Then* $(y_{\alpha}|\alpha\in F)$ *can be found such that* $y \in Y(v)$.
(b) *If* $y \in Y(v)$, *then after arbitrary decreases in any of the* y_{α}, $\alpha \in E$, y *is still in* $Y(v)$.

PROOF.
(a) Take $y_{\alpha} = \max_{\beta\in E} y_{\beta}$, $\alpha \in F$.
(b) The inequalities (5.4) are either unaffected or strengthened by decreasing $(y_{\alpha}|\alpha\in E)$. □

A *ray* for the solution set to a set of linear inequalities is a solution to the corresponding homogeneous set of inequalities (cf. [22]). The rays to $Y(v)$ are therefore the solutions $(y_1,\dots,y_{n^*})$ to:

$$y_{\alpha} \ge \sum_{\beta=1}^{n^*} \tilde{P}^{\xi}_{\alpha\beta} y_{\beta}, \qquad \alpha \in F,\ \xi \in \tilde{K}(\alpha).$$

Define U as the set of rays to Y(v) and remark that U is independent of v, since F, $\tilde{K}(\alpha)$, $\tilde{P}^{\xi}_{\alpha\beta}$ are. Since U is the set of rays to Y(v), it has the following important and easily verified properties:

(a) if $u, \hat{u} \in U$, then $c_1 u + c_2 \hat{u} \in U$ for all $c_1, c_2 \geq 0$

(b) if $v \in V$, $y \in Y(v)$ and $u \in U$, then $y + cu \in Y(v)$ for all $c \geq 0$

REMARK 3. Theorem 5.3 applies to U as well as to Y(v).

Note from theorem 5.2 and theorem 5.3 that the vectors $\bar{u}$ with $\bar{u}_\alpha = cg^{*\alpha}$ and $\bar{\bar{u}}$, with $\bar{\bar{u}}_\alpha = c$, $\alpha \in F$ and $\bar{\bar{u}}_\alpha \leq c$, $\alpha \in E$ are members of U, for any scalar c. Additional properties of U are discussed in theorem 5.4 and section 6.

In order to show that Y(v) is an n^*-dimensional polyhedral set, we need the following discrete time Markovian model with state space $\{1,\ldots,n^*\}$:
For $\alpha \in F$, let $\tilde{K}(\alpha)$ be the set of feasible decision. For $\xi \in \tilde{K}(\alpha)$, let $\tilde{q}^{\xi}_{\alpha}$ and $\tilde{P}^{\xi}_{\alpha\beta}$ denote the associated reward and transition probabilities (we already noted that $\tilde{P}^{\xi}_{\alpha\beta} \geq 0$, $\sum_\beta \tilde{P}^{\xi}_{\alpha\beta} = 1$).
For $\alpha \in E$, add a decision ξ_o to the empty $\tilde{K}(\alpha)$ with $\tilde{q}^{\xi_o}_{\alpha} = -1$ and $\tilde{P}^{\xi_o}_{\alpha\beta} = \delta_{\alpha\beta}$.
Let Φ denote the set of pure policies.
For $\phi \in \Phi$, the quantities $\tilde{q}(\phi)$, $\tilde{P}(\phi)$, $\tilde{\Pi}(\phi)$ and $\tilde{Z}(\phi)$ are defined analogously to q(f), P(F), Π(f) and Z(f) for $f \in S_P$.
Also let $\{\tilde{g}^*_\alpha\}$ be the maximal gain vector for the new process. Note that $\tilde{q}(\phi) \leq 0$ for any $\phi \in \Phi$. The following theorem characterizes the subchains of $\tilde{P}(\phi)$ on F:

THEOREM 5.4. (*Properties of subchains of* $\tilde{P}(\phi)$ *on* F).
Fix $v \in V$. *Suppose for some policy* $\phi \in \Phi$, $\tilde{P}(\phi)$ *has a subchain* $C \subseteq F$. *Then*
(a) C *has at least two members.*
(b) $\tilde{q}(\phi)_\alpha$ *is strictly negative for at least one* $\alpha \in C$.
(c) *There exists a bound* M = M(v) *such that*

$$\max_{\alpha,\beta\in C} |y_\alpha - y_\beta| \leq M \qquad \textit{for any } y \in Y(v).$$

(d) *If* $\bar{y}$ *is a ray to* Y(v) *then* $\bar{y}_\alpha = \bar{y}_\beta$, *for all* $\alpha, \beta \in C$.

PROOF.

(a) Part (a) follows from $\tilde{P}^{\xi}_{\alpha\alpha} < 1$ for any $\alpha \in F$, and $\xi \in \tilde{K}(\alpha)$.

(b) Let policy ϕ use action $(i(\alpha), f(\alpha)) \in \tilde{K}(\alpha)$ for each $\alpha \in C$. For $\alpha \in C$, define $S(\alpha) = \{j \mid P(f(\alpha)^n_{i(\alpha)j} > 0$, for some $n = 0,1,2,\ldots\}$. Note that $i(\alpha) \in S(\alpha)$ and that:

$$(5.6) \qquad \alpha \in C,\ i \in S(\alpha) \text{ imply } P(f(\alpha))_{ij} > 0 \text{ only if } j \in S(\alpha).$$

Now, assume to the contrary that for each $\alpha \in C$, $0 = \tilde{q}(\phi)_\alpha =$ $= Z(f(\alpha))b(v,f(\alpha))_{i(\alpha)}$. Since $f(\alpha) \in S_{PMG}$, $b(v,f(\alpha)) \le 0$ with equality for components in $R(f(\alpha))$. Hence, using (2.3), $0 = \tilde{q}(\phi)_\alpha =$ $= \sum_{j \notin R(f(\alpha))} b(v,f(\alpha))_j = \sum_{j \notin R(f(\alpha))} \sum_{n=0}^{\infty} [P(f(\alpha))]^n_{i(\alpha)j} . b(v,f(\alpha))_j$. Hence:

$$(5.7) \qquad b(v,f(\alpha))_j = 0 \qquad\qquad \text{for } j \in S(\alpha),\ \alpha \in C.$$

We now exhibit a policy $f^o \in S_{RMG}$ with the contradictory properties that $R^o = \cup_{\alpha\in C} [R^{*\alpha} \cup S(\alpha)]$ is closed under $P(f^o)$ while every state in R^o is transient for $P(f^o)$.

Take $f^* \in S^*_{RMG}$. Define f^o as follows:
Initially, for $i \in R^*$ set $\{k \mid f^o_{ik} > 0\} = \{k \mid f^*_{ik} > 0\}$. Then for $i \in S(\alpha)$ add $\{k \mid f(\alpha)_{ik} > 0\}$ to $\{k \mid f^o_{ik} > 0\}$. Finally, for $i \in \Omega \setminus R^o$, set $\{k \mid f^o_{ik} > 0\} = \{k \in L(i) \mid b(v)^k_i = 0\}$.

From (5.7) the definition of f^* in combination with theorem 3.1 part (e), and the definition of f^o on $\Omega \setminus R^o$ it follows that $f^o_{ik} > 0$ implies $b(v)^k_i = 0$, for all i, hence $f^o \in S_{RMG}$.

For $i \in R^o$, (5.6) and the fact that $f^* \in S^*_{RMG}$ imply that $P(f^o)_{ij} > 0$ only for $j \in R^o$; hence, R^o is closed under $P(f^o)$.

As $\sum_{j \notin R^{*\alpha}} \Pi(f(\alpha))_{i(\alpha)j} > 0$, there exist a $j \notin R^{*\alpha}$, and an integer $n \ge 1$, with $P(f(\alpha))^n_{i(\alpha)j} > 0$ and so $P(f^o)^n_{i(\alpha)j} > 0$. Hence $i(\alpha) \in R^{*\alpha}$ is transient under $P(f^o)$, since the subchains of a maximal gain policy are all contained within a single $R^{*\beta}$ (cf. theorem 3.2 part (c)).

Now, observe that for each $\alpha \in C$, all states in $R^{*\alpha}$ communicate with $i(\alpha) \in R^{*\alpha}$ for $P(f^o)$, since they communicate with $i(\alpha)$ for $P(f^*)$. However, this implies that each state in $\cup_{\alpha\in C} R^{*\alpha}$ is transient, since a transient state cannot be reached from a recurrent state.

It remains to prove that each $j \in S(\alpha)$, $(\alpha\in C)$, is transient for $P(f^o)$. Fix $j \in S(\alpha)$, $\alpha \in C$. Since $f(\alpha)$ is maximal gain, there is a state $r \in R^{*\beta}$, for some β, such that $P(f(\alpha))^m_{jr} > 0$, for some $m \geq 1$. Hence $P(f^o)^m_{jr} > 0$. Let n be such that $P(f(\alpha))^n_{i(\alpha)j} > 0$. Finally $\beta \in C$, follows from

$$\tilde{P}(\phi)_{\alpha\beta} \geq \Pi(f(\alpha))_{i(\alpha)r}' = [P(f(\alpha))^n\Pi(f(\alpha))]_{i(\alpha)r} \geq$$

$$\geq P(f(\alpha))^n_{i(\alpha)j}\ \Pi(f(\alpha))_{jr} > 0$$

and the fact that C is a subchain of $\tilde{P}(\phi)$. This implies that r is transient for $P(f^o)$ and so is j, since a transient state cannot be reached from a recurrent state.

(c) Introduce a slack vector $t \geq 0$ and rewrite (5.4) as:

$$y = \tilde{q}(\phi) + t + \tilde{P}(\phi)y. \tag{5.8}$$

Let $\{\tilde{\pi}^C(\phi)_\alpha \mid \alpha \in C\}$ denote the unique equilibrium distribution of $\tilde{P}(\phi)$ on C. Multiply (5.8) with $\tilde{Z}(\phi)$. Then, since $\tilde{Z}(\phi)_{\beta\gamma} = 0$ for $\beta \in C$, $\gamma \notin C$ (cf. (2.3)):

$$y_\beta = \sum_{\gamma\in C} \tilde{Z}(\phi)_{\beta\gamma}(\tilde{q}(\phi)_\gamma + t_\gamma) + \sum_{\gamma\in C} \tilde{\pi}^C(\phi)_\gamma y_\gamma, \qquad \text{all } \beta \in C$$

Part (c) follows with the choice $M = 2\max_{\beta\in C}\{\sum_{\alpha\in C} |\tilde{Z}(\phi)_{\beta\alpha}|\,[|\tilde{q}(\phi)_\alpha| + t_\alpha]\}$ provided one shows that $[t_\alpha \mid \alpha \in C]$ are bounded uniformly in y. However, by multiplying (5.7) with $\tilde{\pi}^C(\phi)$ one obtains:

$$-\sum_{\beta\in C} \tilde{\pi}^C(\phi)_\beta \tilde{q}(\phi)_\beta = \sum_{\beta\in C} \tilde{\pi}^C(\phi)_\beta t_\beta.$$

The boundedness of $[t_\beta \mid \beta \in C]$ follows since $\tilde{\pi}^C(\phi)_\beta > 0$ for $\beta \in C$.

(d) Use part (c) and (5.5). □

Together part (b) of theorem 5.4 and the choice $\tilde{q}_\alpha^{\xi_0} = -1$, for $\alpha \in E$ imply:

<u>COROLLARY 5.1.</u> $\tilde{g}^*_\alpha < 0$ *for* $\alpha = 1,\dots,n^*$.

THEOREM 5.5. (*Cf. theorem* 3 *of* [15].) *Fix* $v \in V$. *Given any* $\{y_\alpha \mid \alpha \in E\}$ *there exist* $\{y_\alpha \mid \alpha \in F\}$ *such that*

$$y_\alpha > \tilde{q}_\alpha^\xi + \sum_{\beta=1}^{n^*} \tilde{P}_{\alpha\beta}^\xi y_\beta, \qquad \textit{for all } \alpha \in F,\ \xi \in \tilde{K}(\alpha) \tag{5.9}$$

holds with strict inequality.

PROOF. It suffices to show that there exists a solution y^o to (5.9) for some $\{y_\alpha^o \mid \alpha \in E\}$ since a solution for any $\{y_\alpha \mid \alpha \in E\}$ is then obtained by adding a ray u with $u_\alpha = y_\alpha - y_\alpha^o$, for $\alpha \in E$ (cf. remark 3).

Since $\tilde{q}_\alpha^{\xi_o} = -1$ and $\tilde{P}_{\alpha\alpha}^{\xi_o} = 1$, for $\alpha \in E$, the solution set to (5.9) is not altered by adding the inequalities $y_\alpha > \tilde{q}_\alpha^{\xi_o} + \sum_{\beta=1}^{n^*} \tilde{P}_{\alpha\beta}^{\xi_o} y_\beta$, $\alpha \in E$. Now, assume to the contrary, that the solution set of (5.9) is empty. Then for the LP-problem:

$$\min Z \quad \text{subject to}$$

$$y_\alpha + Z \geq \tilde{q}_\alpha^\xi + \sum_{\beta=1}^{n} \tilde{P}_{\alpha\beta}^\xi y_\beta, \qquad \alpha = 1,\ldots,n^*;\ \xi \in \tilde{K}(\alpha),$$

we have min $Z \geq 0$, which according to theorem 2 of [14], implies $\max_{\alpha=1,\ldots,n^*} \tilde{g}_\alpha^* \geq 0$. This contradicts corollary 5.1. □

Since the solution set to (5.9) is open, for any y satisfying (5.9), there exists a $\delta > 0$, so that $|y - y'| < \delta$ implies $y' \in Y(v)$. Hence the n^* parameters $(y_1,\ldots,y_{n^*})$ may be chosen independently over some (finite) region. V and V^R have exactly $n^* = \|E \cup F\|$ degrees of freedom, of which $\|E\|$ are globally independent and $\|F\|$ are only locally independent.

VI. TRIANGULAR DECOMPOSITION OF Y(v)

Define the following partition of F:

$F^\ell = \{\alpha \in F \mid$ for every $\phi \in \Phi$, α reaches E with certainty under $\tilde{P}(\phi)\}$

$F^t = \{\alpha \in F \mid \alpha$ is transient under any $\tilde{P}(\phi)$, $\phi \in \Phi$, but $\alpha \notin F^\ell\}$

$F^r = \{\alpha \in F \mid \alpha$ is recurrent for some $\tilde{P}(\phi)$, $\phi \in \Phi\}$.

Note that $\sum_{\beta \in F^\ell \cup E} \tilde{P}_{\alpha\beta}^\xi = 1$ for $\alpha \in F^\ell$, $\xi \in \tilde{K}(\alpha)$.

The set of inequalities (5.2) then decouples into 3 parts:

$$(6.1)\qquad y_\alpha \geq \Big[\tilde{q}^\xi_\alpha + \sum_{\beta\in E\cup(F\setminus F^t)} \tilde{P}^\xi_{\alpha\beta} y_\beta\Big] + \sum_{\beta\in F^t} \tilde{P}^\xi_{\alpha\beta} y_\beta, \qquad \alpha\in F^t,\ \xi\in\tilde{K}(\alpha)$$

$$(6.2)\qquad y_\alpha \geq \Big[\tilde{q}^\xi_\alpha + \sum_{\beta\in E} \tilde{P}^\xi_{\alpha\beta} y_\beta\Big] + \sum_{\beta\in F^\ell} \tilde{P}^\xi_{\alpha\beta} y_\beta, \qquad \alpha\in F^\ell,\ \xi\in\tilde{K}(\alpha)$$

$$(6.3)\qquad y_\alpha \geq \Big[\tilde{q}^\xi_\alpha + \sum_{\beta\in E\cup(F\setminus F^r)} \tilde{P}^\xi_{\alpha\beta} y_\beta\Big] + \sum_{\beta\in F^r} \tilde{P}^\xi_{\alpha\beta} y_\beta, \qquad \alpha\in F^r,\ \xi\in\tilde{K}(\alpha).$$

The above decomposition implies that the following vectors belong to U: $u_\alpha = c_1$, $\alpha\in E$; $u_\alpha = c_2$, $\alpha\in F^\ell$; $u_\alpha = c_3$, $\alpha\in F^t\cup F^r$; for all c_1,c_2,c_3 with $c_1\leq c_2\leq c_3$. For $\phi\in\Phi$, let $W(\phi) = [P(\phi)_{\alpha\beta}]_{\alpha,\beta\in F^\ell\cup F^t}$.

Then $W(\phi)$ is a substochastic transient matrix, with $\lim_{n\to\infty} W(\phi)^n = 0$ and $[I - W(\phi)]^{-1} = \sum_{n=0}^\infty W(\phi)^n$ exists and is non-negative. Then, taking together (6.1) and (6.2) and using the proof of lemma 1 of [7], we obtain:

$$(6.4)\qquad y_\alpha \geq \max_{\phi\in\Phi} \sum_{\beta\in F^\ell\cup F^t} [I - W(\phi)]^{-1}_{\alpha\beta}\, [\tilde{q}(\phi)_\beta + \sum_{\gamma\in E\cup F^r} \tilde{P}(\phi)_{\beta\gamma} y_\gamma],$$

$$\alpha\in F^t\cup F^\ell.$$

Insert (6.4) into (6.3) in order to obtain:

$$(6.5)\qquad y_\alpha \geq \hat{q}^{\xi,\phi}_\alpha + \sum_{\beta\in E\cup F^r} \hat{P}^{\xi,\phi}_{\alpha\beta}\, y_\beta, \qquad \text{all } \alpha\in F^r,\ \xi\in\tilde{K}(\alpha),\ \phi\in\Phi,$$

where

$$\hat{q}^{\xi,\phi}_\alpha = \tilde{q}^\xi_\alpha + \sum_{\beta\in F^\ell\cup F^t} \tilde{P}^\xi_{\alpha\beta} \sum_{\gamma\in F^\ell\cup F^t} [I - W(\phi)]^{-1}_{\beta\gamma}\, \tilde{q}(\phi)_\gamma$$

$$\hat{P}^{\xi,\phi}_{\alpha\beta} = \tilde{P}^\xi_{\alpha\beta} + \sum_{\gamma\in F^\ell\cup F^t} \tilde{P}^\xi_{\alpha\gamma} \sum_{\delta\in F^\ell\cup F^t} [I - W(\phi)]^{-1}_{\gamma\delta}\, \tilde{P}(\phi)_{\delta\beta}.$$

Notice that $\hat{q}^{\xi,\phi}_\alpha \leq 0$, and $\hat{P}^{\xi,\phi}_{\alpha\beta} \geq 0$ with $\sum_{\beta\in E\cup F^r} \hat{P}^{\xi,\phi}_{\alpha\beta} = 1$.

Observe that (6.5) relates $\{y_\alpha \mid \alpha\in F^r\}$ to $\{y_\alpha \mid \alpha\in E\}$, and remark that (6.5) always has a solution $\{y_\alpha \mid \alpha\in F^r\}$ no matter how $\{y_\alpha \mid \alpha\in E\}$ are specified (take $y_\alpha = \max_{\beta\in E} y_\beta$, for all $\alpha\in F^r$).

<u>THEOREM 6.1</u>. *Fix* $v\in V$.

(a) *If* $y\in Y(v)$, *i.e. if* y *satisfies* (6.1), (6.2), (6.3) *it satisfies* (6.5) *as well.*

(b) *Conversely, if one picks* $\{y_\alpha \mid \alpha \in E\}$ *arbitrarily, next picks* $\{y_\alpha \mid \alpha \in F^r\}$ *to satisfy* (6.5), *next defines* $\{y_\alpha \mid \alpha \in F^t \cup F^\ell\}$ *as the right-hand side of* (6.4), *then the resulting vector* $\{y_\alpha \mid \alpha \in E \cup F\}$ *satisfies* (6.1), (6.2), (6.3), *hence belongs to* $Y(v)$.

PROOF.

Part (a) follows from the above remarks.

(b) Observe that the right-hand side of (6.4) may be interpreted as the maximal total expected return of a terminating discrete-time Markovian model, with $F^t \cup F^\ell$ as state space. Because of the choice:

$$(6.6)\qquad y_\alpha = \max_{\phi\in\Phi} \sum_{\beta\in F^\ell \cup F^t} [I - W(\phi)]^{-1}_{\alpha\beta} [\tilde{q}(\phi)_\beta + \sum_{\gamma\in E\cup F^r} \tilde{P}(\phi)_{\beta\gamma} y_\gamma], \quad \text{for } \alpha \in F^t \cup F^\ell,$$

it hence follows from corollary 2 of [21] that $y_\alpha = \tilde{q}(\phi)_\alpha + \sum_{\beta\in E\cup F^r} \tilde{P}(\phi)_{\alpha\beta} y_\beta + \sum_{\beta\in F^t\cup F^\ell} W(\phi)_{\alpha\beta} y_\beta$, $\alpha \in F^\ell$. Hence, the vector y satisfies (6.1) and (6.2)

In addition, using corollary 1 of [21], it follows that there exists a $\phi^* \in \Phi$ that maximizes the right-hand side of (6.6) simultaneously for all $\alpha \in F^t \cup F^\ell$, given any $\{y_\alpha \mid \alpha \in E \cup F^r\}$. Consider the inequalities (6.5) for $\phi = \phi^*$, and use (6.6) in order to show that the vector y satisfies (6.3) as well. □

REMARK 4. This provides a triangular decomposition in that one first determines $\{y_\alpha \mid \alpha \in E\}$, next $\{y_\alpha \mid \alpha \in F^r\}$ and finally $\{y_\alpha \mid \alpha \in F^\ell \cup F^t\}$. The last part can actually be decomposed further, by first determining $\{y_\alpha \mid \alpha \in F^\ell\}$ and then determining $\{y_\alpha \mid \alpha \in F^t\}$ via

$$y_\alpha = \max_{\phi\in\Phi} \sum_{\beta\in F^\ell} [I - W(\phi)^\ell]^{-1}_{\alpha\beta} [\tilde{q}(\phi)_\beta + \sum_{\gamma\in E} \tilde{P}(\phi)_{\beta\gamma} y_\gamma], \qquad \alpha \in F^\ell$$

$$y_\alpha = \max_{\phi\in\Phi} \sum_{\beta\in F^t} [I - W(\phi)^t]^{-1}_{\alpha\beta} [\tilde{q}(\phi)_\beta + \sum_{\gamma\in E\cup F^\ell \cup F^r} \tilde{P}(\phi)_{\beta\gamma} y_\gamma], \quad \alpha \in F^t,$$

where the transient matrices $W(\phi)^\ell$ and $W(\phi)^t$ are defined by:

$$W(\phi)^\ell \equiv [\tilde{P}(\phi)_{\alpha\beta}]_{\alpha,\beta\in F^\ell}; \qquad W(\phi)^t = [\tilde{P}(\phi)_{\alpha\beta}]_{\alpha,\beta\in F^t}.$$

Example 2 below has $N = 7$, $g_i^* = 0$, $L(i) = K(i)$ for all i

$R^* = \cup_{i=1}^{7} R^{*i}$ with $R^{*i} = \{i\}$, i.e. $n^* = 7$

$E = \{\alpha = 1\}$; $F^{\ell} = \{\alpha = 4\}$; $F^{t} = \{\alpha = 7\}$; $F^{r} = \{\alpha = 2,3,5,6\}$.

V is the solution set to the following decomposed set of inequalities:

$\alpha = 1$: v_1 arbitrary

$\alpha = 4$: $v_4 \geq q_4^2 + v_1$

$\alpha = 7$: $v_7 \geq q_7^2 + .5(v_1+v_2)$

$\alpha = (2,3)$: $q_2^2 \leq v_2 - v_3 \leq -q_3^2$

$\alpha = (5,6)$: $v_5 \geq q_5^2 + v_6,\ q_5^2 + q_6^3 + .5(v_1+v_2),\ q_5^2 + q_6^3 + .5q_2^2 + .r(v_1+v_3)$

$v_6 \geq q_6^2 + v_5,\ q_6^3 + .5(v_1+v_2),\ q_6^3 + .5q_2^2 + .5(v_1+v_2)$.

Example 2

i	k	q_i^k	p_{i1}^k	p_{i2}^k	p_{i3}^k	p_{i4}^k	p_{i5}^k	p_{i6}^k	p_{i7}^k
1	1	0	1						
2	1	0		1	0				
	2	0		0	1				
3	1	0		0	1				
	2	0		1	0				
4	1	0				1			
	2	0	1						
5	1	0					1	0	
	2	0					0	1	
6	1	0					0	1	
	2	0					1	0	
	3	0	.5	.5					
7	1	0							1
	2	0	.5	.5					

Absent p_{ij}^k are zero.

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